2020

MATHEMATICS — HONOURS

Fifth Paper

(Module - IX)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

($\mathbb{Q}$, $\mathbb{R}$, $\mathbb{N}$ denote the sets of rational numbers, real numbers and natural numbers respectively)

Answer ***question no.*** **1** and ***any two*** questions from the rest.

1. (a) Answer ***any one*** of the following : 6

(i) Prove or disprove total variation of $\sin x + \cos x$ on $\left[0, \frac{\pi}{4}\right]$ is $\sqrt{2}$.

(ii) Correct or justify : A Riemann-integrable function f on $[a, b]$ may be neither continuous nor monotone on $[a, b]$.

(iii) Find the limit function of the sequence $\{f_n\}$ given by $f_n(x) = \frac{[nx]}{n}, 0 \le x \le 1;\ n \in \mathbb{N}$.

([y] denote the largest integer less than or equal to y).

(iv) Prove or disprove : The power series $\sum_{n=0}^{\infty} a_n x^n$ and $\sum_{n=1}^{\infty} n.a_n x^{n-1}$ have same radius of convergence.

(b) Prove or disprove ***any one*** of the following : 4

(i) The set $A = \left\{5 + x\sqrt{2} : x \in \mathbb{Q}\right\}$ is of measure zero.

(ii) Every bounded enumerable set is compact.

(iii) The function $f(x) = \sum_{n=1}^{\infty} \frac{\sin\left(n^2 x\right)}{n^2}$ is continuous on $\mathbb{R}$.

(iv) Radius of convergence of $1 + \frac{x}{2} + \left(\frac{x}{4}\right)^2 + \left(\frac{x}{2}\right)^3 + \left(\frac{x}{4}\right)^4 + ...$ is 2.

2. (a) If S is a bounded, closed subset of $\mathbb{R}$, prove that every infinite open cover of S has a finite subcover.

(b) Choosing a suitable open cover, prove that $A = (0, 1) \cup \{5, 6\}$ is not compact.

(c) If a function f is Riemann-integrable on $[a, b]$, prove that the set

$$S = \left\{ x \in [a, b] \Big/ \int_x^b f(t)\,dt \text{ is continuous} \right\} \text{ is compact.}$$

10+6+4

3. (a) Construct a real valued function on a compact interval which is uniformly continuous but not of bounded variation on that interval.

(b) If $f:[a, b] \to \mathbb{R}$ is of bounded variation on $[a, b]$, prove that its variation function is monotonically increasing on $[a, b]$.

(c) Let $f, g:[0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x^2 \cos\frac{\pi}{x^2} & \text{if } x \in (0,1] \\ 0 & \text{if } x = 0 \end{cases}$

and $g(x) = e^{x^2+1}$, $x \in [0,1]$.

Examine whether $\gamma = (f, g)$ is rectifiable.

6+6+8

4. (a) Prove that a bounded function $f:[a, b] \to \mathbb{R}$ is Riemann Integrable on $[a, b]$ if and only if for every $\varepsilon\,(> 0)$ there is a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \varepsilon$.

(b) If f, g are Riemann integrable on $[a, b]$ and $|g(x)| > 1$ for all $x \in [a, b]$, use Lebesgue's criterion to show that $\frac{f}{g}$ is Riemann integrable on $[a, b]$.

(c) If $f:[a, b] \to \mathbb{R}$ is a continuous function such that $\int_a^b f^2(x)dx = 0$ then prove that the set $\{x \in [a, b] / f(x) = 0\}$ is uncountable.

8+6+6

5. (a) If $f:[a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$, prove that $\int_a^b f(x)\,dx = \mu(b-a)$,

where $\inf\limits_{x\in[a,b]} f(x) \le \mu \le \sup\limits_{x\in[a,b]} f(x)$.

(b) Correct or justify : If a real valued function f has a primitive on $[a, b]$, then f is Riemann integrable on $[a, b]$.

(c) Prove that $\frac{\pi}{6} > \int_0^{1/2} \frac{dx}{\sqrt{1-x^{2020}}} > \frac{1}{2}$.

6+6+8

6. (a) Let $\{f_n\}$ be a sequence of functions defined on $[a, b]$ such that $\lim_{n\to\infty} f_n(x) = f(x), \forall x \in [a, b]$ and

$$M_n = \sup_{x\in[a,b]} |f_n(x) - f(x)|.$$

Show that $\{f_n\}$ converges uniformly to f on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$.

(b) Let $f_n(x) = \begin{cases} \frac{x}{n^2} & \text{if } n \text{ is even} \\ \frac{1}{n^2} & \text{if } n \text{ is odd} \end{cases}$ where $x \in \mathbb{R}$.

Find the limit function of $\{f_n\}$ with proper justification. Is the convergence uniform? Justify.

(c) Let f be a real valued uniformly continuous function on $\mathbb{R}$. If $f_n(x) = f\left(x + \frac{1}{n}\right)$ for all $x \in \mathbb{R}$, for all $n \in \mathbb{N}$, then prove that $\{f_n\}$ is uniformly convergent on $\mathbb{R}$. 8+(4+2)+6

7. (a) Prove that the sum function of a uniformly convergent series $\sum_n f_n$ of continuous functions defined on a set $D \subseteq \mathbb{R}$ is continuous on D.

(b) Examine whether $\sum_{n=1}^{\infty}\left[n^2 x e^{-n^2x^2} - (n-1)^2 x e^{-(n-1)^2 x^2}\right]$ is uniformly convergent on [0, 1].

(c) Using Abel's test prove that the series $\sum_{n=1}^{\infty} a_n n^{-x}$ converges uniformly on [0, 1] if $\sum_{n=1}^{\infty} a_n$ converges uniformly on [0, 1]. 8+8+4

8. (a) If a power series $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $\rho \in (0, \infty)$ and $\sum_{n=0}^{\infty} a_n \rho^n$ is convergent, prove that the power series is uniformly convergent on $[0, \rho]$.

(b) Find the largest interval in which the power series $\sum_{n=1}^{\infty} \frac{x^n}{n.10^{n-1}}$ is convergent.

(c) Prove or disprove : If a power series is neither nowhere convergent nor everywhere convergent, then its sum function is bounded. 8+8+4

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